Proposition 1 *If $n \times n$ matrices A, B and $A - B$ are positive definite, then $B^{-1} - A^{-1}$ is also positive definite.*

Proof. (by Kaiming Zhao, with an assist to Ed Wang, both of WLU.) Note first that A and B are hermitian ($A = A^*$ and $B = B^*$). Then $B^{-1} - A^{-1}$ is pd if and only if $A(B^{-1} - A^{-1})A^*$ is pd. But

$$\begin{aligned}
A(B^{-1} - A^{-1})A^* &= A(B^{-1} - A^{-1})A \\
&= (AB^{-1} - I)A \\
&= (((A - B) + B)B^{-1} - I)A \\
&= (A - B)B^{-1}A \\
&= (A - B)B^{-1}(A - B + B) \\
&= (A - B)B^{-1}(A - B) + (A - B) \\
&= (A - B)B^{-1}(A - B)^* + (A - B)
\end{aligned}$$

is pd because $A - B$ and B^{-1} are pd. ∎

Facts used in the proof (supplied by Ed):

- if K is pd, then so is K^{-1} (eigenvalues are reciprocals)
- if H and K are both pd, then so is $H + K$ ($\langle (H + K)x, x \rangle > 0$ using linearity)
- if K is pd, then H is pd iff KHK is pd (play with $\langle KHKx, x \rangle$)